arXiv:gr-qc/0112033v1 16 Dec 2001

Conformal Killing Vectors in Spherically Symmetric, Inhomogeneous, Shear-free, Separable Metric Spacetimes

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December 10, 2001

Abstract

In this paper, we find all the Conformal Killing Vectors (CKVs) and their Lie Algebra for the recently reported [1] spherically symmetric, shear-free, separable metric spacetimes with non-vanishing energy or heat flux. We also solve the geodesic equations of motion for the spacetime under consideration.

To be submitted: Classical and Quantum Gravity

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1 Introduction

In a recent paper [1], all the spherically symmetric, inhomogeneous, shear-free, separable-metric spacetimes, that could admit matter with an equation of state $p = \alpha\rho$ where p is the pressure, ρ is the density of matter in the spacetime and α is a constant, were reported. These spacetimes could be used to model a shear-free spherical star [2] or the shear-free universe at large [3]. It is of some definite interest, and, therefore, it is the purpose of this paper, to investigate the Conformal Killing Vectors (CKVs) of these spacetimes. We also solve the corresponding geodesic equations.

The organization of this paper is as follows. In §2, we provide the salient features of the spacetimes of [1]. In §3, we obtain all the CKVs for these spacetimes. In particular, we obtain the Killing vectors in §3.1; the Homothetic Killing vectors in §3.2 and the Non-Special Conformal Killing vectors in §3.4. There are no Special Conformal Killing vectors for these spacetimes as pointed out in §3.3. A summary of these results is provided in §4 along with the table for conformal factors for each CKV. Also provided in this section are the structure constants of the Lie Algebra of the CKVs. In §5, we solve the geodesic equations of motion for the spacetime under consideration. In the concluding section, §6, we discuss the physical applications.

2 The Spacetime

The metric of all the spherically symmetric, inhomogeneous, shear-free, separable-metric spacetimes, that admit matter with an equation of state $p = \alpha\rho$ where p is the pressure, ρ is the density of matter in the spacetime and α is a constant, is :

$$ds^2 \;=\; -\,y^2(r)\,dt^2 \;+\; 2\,R^2(t)\left(\frac{dy}{dr}\right)^2 dr^2 \;+\; y^2(r)\,R^2(t)\left[\,d\theta^2 \;+\; \sin^2\theta\,d\phi^2\,\right] \qquad (1)$$

It should be noted that the metric (1), in general, has [1]: $g_{tt} = -\,y^2\,A^2(t)$. However, a redefinition of the time coordinate can be made to absorb the temporal function $A(t)$ without affecting any physical or geometrical properties.

In what follows, we shall denote a time derivative by an overhead dot and derivative with respect to the radial coordinate by an overhead prime. Here (t, r, θ, ϕ) are the co-

moving coordinates. In a Lorentz frame, this metric has the Einstein tensor

$$G_{tt} = \frac{1}{2y^2R^2} + \frac{3\dot{R}^2}{y^2R^2} \tag{2}$$

$$G_{tr} = \frac{\sqrt{2}\dot{R}}{y^2R^2} \tag{3}$$

$$G_{rr} = G_{\theta\theta} = G_{\phi\phi} = \frac{1}{2y^2R^2} - \frac{2\ddot{R}}{y^2R} - \frac{\dot{R}^2}{y^2R^2} \tag{4}$$

Note that the G_{tr} component of the Einstein tensor is non-vanishing for $\dot{R} \neq 0$. The presence of the G_{tr} component of the Einstein tensor implies the presence of an energy flux across $r =$ constant surfaces in the spacetime of metric (1). This metric admits the following invariants:

$$R = \frac{1 - 6\dot{R}^2 - 6R\ddot{R}}{R^2y^2} \tag{5}$$

$$R_{ab}R^{ab} = \frac{1 + 12\dot{R}^4 - 6R\ddot{R} + 12R^2\ddot{R}^2 - 4\dot{R}^2(1 - 3R\ddot{R})}{R^4y^4} \tag{6}$$

$$R_{abcd}R^{abcd} = \frac{3 - 4\dot{R}^2 + 12\dot{R}^4 - 8R\ddot{R} + 12R^2\ddot{R}^2}{R^4y^4} \tag{7}$$

Clearly (1) is singular iff $y(r) = 0$ for some r and/or $R(t) = 0$ for some t.

The fluid four-velocity U^a is co-moving and is given by

$$U^a = \frac{1}{y}\delta^a_0 \tag{8}$$

The fluid four-acceleration $\dot{U}_a = U_{a;b}U^b$ is given by

$$\dot{U}_a = (0, \frac{y'}{y}, 0, 0) \tag{9}$$

where a semicolon denotes a covariant derivative. The spacetime under consideration has non-zero acceleration unless $y' = 0$.

The presence of the energy flux implies[1] the energy-momentum tensor of the matter

[1] While a non-vanishing radial heat flow is one possible interpretation, it is not unique. All one really knows is that there must be radial energy transport. This could be caused by heat, or could just as well be caused by the bulk motion of matter or radiation.

fluid to be of the form

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} + Q_a U_b + Q_b U_a \tag{10}$$

where p is the pressure, ρ is the density of the fluid and $Q^a = (0, Q, 0, 0)$ is the radial energy or the radial heat flux four-vector. Note that both shear and rotation vanish for the metric (1).

The expansion, Θ, energy or heat flux, Q, density, ρ and pressure, p, are given by

$$\Theta = U^a{}_{;a} = \frac{3}{y}\frac{\dot{R}}{R} \tag{11}$$

$$Q = \frac{-1}{y^2\, y'}\frac{\dot{R}}{R^3} \tag{12}$$

$$\rho = \frac{1}{y^2\, R^2}\left[\frac{1}{2} + 3\,\dot{R}^2\right] \tag{13}$$

$$p = \frac{1}{2y^2 R^2} - \frac{\dot{R}^2}{y^2 R^2} - \frac{2\ddot{R}}{y^2 R} \tag{14}$$

From eqs. (13) and (14), we obtain

$$\ddot{R} = \frac{y^2 R}{6}\left[\frac{2}{y^2 R^2} - (\rho + 3\,p)\right] \tag{15}$$

Therefore, the equation of state, that is, a relation between p and ρ, of the matter in the spacetime of (1) determines the dynamics - the behavior of the temporal function $R(t)$ - of the spacetime. The temporal behavior of the heat flux is also determined by the temporal function $R(t)$.

The spatial or radial nature of the heat flux is determined primarily by the sign of the quantity $-\,\dot{R}/y'$. The heat flux is positive, that is, heat flows from lower values of r to higher values of r, when y' and $\dot{R}$ have opposite signs. In the case of these two quantities having the same sign, the heat flux is negative, that is, heat flows from higher values of r to lower values of r. On the other hand, the density is, for $y' > 0$, a decreasing function

of r corresponding to a region over-dense at its center and, for $y' < 0$, an increasing function of r corresponding to a region under-dense at its center.

For $\dot{R} > 0$ ($\dot{R} < 0$), the spacetime under consideration is expanding (contracting) since the expansion is positive (negative), $\Theta > 0$ ($\Theta < 0$). For $\dot{R} > 0$, the heat flux is positive for $y' < 0$ and negative for $y' > 0$. Therefore, when the spacetime is expanding, heat flows from under-dense regions to over-dense regions. For $\dot{R} < 0$, the heat flux is positive for $y' > 0$ and negative for $y' < 0$. Therefore, when the spacetime is contracting, heat flows from over-dense regions to under-dense regions. Any two adjacent regions with opposite signs for heat flux are joined by heat flow caustics at locations for which $y' = 0$. Note that the quantity Q blows up at such locations.

However, we note that the density is finite at locations $y' = 0$. The density is infinite only at locations $y(r) = 0$. We also observe that the metric (1) is singular at heat flow caustics *i.e.* for $y' = 0$. Such locations are, however, coordinate singularities since the curvature invariants (5), (6) and (7) are finite there.

We also observe here that the radial function $y(r)$ is not determined by the field equations. Therefore, radial attributes of matter are *arbitrary*, meaning, unspecified, for the metric (1). This is in the manner of concentric spheres with each sphere allowed to possess any value of density, for example. This is the maximal freedom compatible with the assumption of spherical symmetry, we may note.

The point $r = 0$ will possess a locally flat neighborhood when $y'|_{r\sim 0} \approx 1/\sqrt{2}$. This condition must be imposed on any $y(r)$. Apart from this condition, the function $y(r)$ is arbitrary. Other physical considerations, for example such as those arising from the equation of heat transfer in the spacetime, could also constrain the function $y(r)$.

A stellar model with heat flux can be obtained for an appropriate choice of the radial function $y(r)$ [1]. On the other hand, the metric (1) can also provide an inhomogeneous cosmological model with non-vanishing heat flow.

This essentially completes our description of the spherically symmetric, inhomogeneous, shear-free, separable-metric spacetimes that could admit an equation of state $p = \alpha\,\rho$ for the matter in the spacetime [1].

3 Conformal Killing Vectors

Conformal symmetries are of some importance in the understanding of spacetime geometry. The conformal motions preserve the angles between vectors and, hence, the light cone structure of the spacetime. These symmetries thereby help solve the geodesic equations of motion for the spacetime under consideration.

A Conformal Killing Vector (CKV) $\mathbf{X}$ satisfies

$$\mathcal{L}_{\mathbf{X}}\, g_{ij} \;=\; 2\,\Phi(x^k)\, g_{ij} \tag{16}$$

where $\Phi(x^k)$ is the conformal factor and g_{ij} is the spacetime metric tensor. There typically arise the following four special cases of CKVs, namely,

1]	Killing vectors	K_J	$\Phi = 0$
2]	Homothetic Killing vectors	H_J	$\Phi_{,j} = 0 \neq \Phi$
3]	Special Conformal Killing vectors	S_J	$\Phi_{,ij} = 0 \neq \Phi_{,j} \neq \Phi$
4]	Non-Special Conformal Killing vectors	N_J	$\Phi_{,ij} \neq 0 \neq \Phi_{,j} \neq \Phi$

These vectors are of physical significance as they help produce first integrals. In particular, the Killing vectors generate the constants of motion and the homothetic Killing vectors scale distances by the same constant factor and, hence, preserve the null geodesic affine parameters. Conformal Killing vectors generate constants of motion along null geodesics. For details, see [4], [5].

The set of Conformal Killing Vectors forms a Lie Algebra G_r ($r \leq 15$) with the basis $\{X_l\}$: $\mathcal{L}_{X_l}\, g_{ij} \;=\; 2\,\Phi_l\, g_{ij}$ such that

$$[X_l,\, X_j] \;=\; C^k_{lj} X_k \qquad\qquad X_l \Phi_j \;-\; X_j \Phi_l \;=\; C^k_{lj} \Phi_k \tag{17}$$

The eq. (16) for the metric (1) reduces to the system

$$(t,t) \qquad \frac{y'}{y} X^r \;+\; X^t{}_{,t} \;=\; \Phi \tag{18}$$

$$(t,r) \qquad -\,y^2\, X^t{}_{,r} \;+\; 2(y')^2 R^2 X^r{}_{,t} \;=\; 0 \tag{19}$$

$$(t,\theta) \qquad -\,X^t{}_{,\theta} \;+\; R^2 X^\theta{}_{,t} \;=\; 0 \tag{20}$$

$$
\begin{array}{llll}
(t,\phi) & -X^t{}_{,\phi} + R^2\sin^2\theta X^\phi{}_{,t} & = 0 & (21)\\
(r,r) & \frac{\dot R}{R}X^t + \frac{y''}{y'}X^r + X^r{}_{,r} & = \Phi & (22)\\
(r,\theta) & y^2 X^\theta{}_{,r} + 2(y')^2 X^r{}_{,\theta} & = 0 & (23)\\
(r,\phi) & y^2\sin^2\theta X^\phi{}_{,r} + 2(y')^2 X^r{}_{,\phi} & = 0 & (24)\\
(\theta,\theta) & \frac{\dot R}{R}X^t + \frac{y'}{y}X^r + X^\theta{}_{,\theta} & = \Phi & (25)\\
(\theta,\phi) & X^\theta{}_{,\phi} + \sin^2\theta X^\phi{}_{,\theta} & = 0 & (26)\\
(\phi,\phi) & \frac{\dot R}{R}X^t + \frac{y'}{y}X^r + \cot\theta X^\theta + X^\phi{}_{,\phi} & = \Phi & (27)
\end{array}
$$

In what follows, we obtain the Conformal Killing Vectors from the above equations case by case.

3.1 Killing vectors

As can be easily verified, the metric (1) admits the following three spacelike Killing vectors:

$$X_I \equiv K_I = (0,\,0,\,0,\,1) \tag{28}$$

$$X_{II} \equiv K_{II} = (0,\,0,\,\sin\phi,\,\cos\phi\cot\theta) \tag{29}$$

$$X_{III} \equiv K_{III} = (0,\,0,\,\cos\phi,\,-\sin\phi\cot\theta) \tag{30}$$

Now, for any timelike Killing vector $K_{IV} = (\,K^t,\,0,\,0,\,0\,)$ orthogonal to $t =$ constant hyper-surfaces, eqs. (19) - (27) are identically satisfied. The eq. (18) then implies

$$X_{IV} \equiv K_{IV} = (1,\,0,\,0,\,0) \qquad \text{condition } \dot R = 0 \tag{31}$$

Now, for a Killing vector $K_V = (\,K^t(t),\,K^r(r),\,0,\,0\,)$ that is *not* hyper-surface orthogonal, eqs. (19), (20), (21), (23), (24), (26) can be satisfied. Then, eqs. (18), (22), (25), (27) imply

$$X_V \equiv K_V = \left(\kappa\,R,\,-\frac{y}{y'}\,\kappa\,\mathcal{C},\,0,\,0\right) \qquad \text{condition } \ddot R = 0 \tag{32}$$

where κ is a constant and $\dot R = \mathcal{C}$.

The Killing vectors K_I, K_{II} and K_{III} are always spacelike, K_{IV} is always timelike and K_V can be spacelike, null or timelike. Any spherically symmetric spacetime admits K_I - K_{III} as spacelike Killing vectors. The conformal factor in each of these cases is zero.

We note that the spacetime admitting K_{IV} as a Killing vector is a *static* spacetime. The Killing vector K_V is neither normal nor tangent to t = constant hyper-surfaces. It is known [6] that conformal motions generated by conformal Killing vectors do not, in general, map a fluid flow conformally.

3.2 Homothetic Killing vectors

In this case, Φ = constant.

For $H_I = (0,\, H^r,\, 0,\, 0)$, eqs. (19), (20), (21), (23), (24), (26) can be satisfied. Eqs. (18), (22), (25), (27), then, imply

$$X_{VI} \equiv H_I = \left(0,\, B_I\,\frac{y}{y'},\, 0,\, 0\right) \tag{33}$$

For $H_{II} = (H^t,\, 0,\, 0,\, 0)$, eqs. (19), (20), (21), (23), (24), (26) can be satisfied. Eqs. (18), (22), (25), (27) yield

$$X_{VII} \equiv H_{II} = \left(B_{II}\,\frac{R}{\dot{R}},\, 0,\, 0,\, 0\right) \qquad \text{condition } \ddot{R} = 0 \tag{34}$$

For $H_{III} = \left(H^t,\, 0,\, 0,\, H^\phi\right)$, eqs. (19), (20), (23), (24), (26) can be satisfied. Eqs. (18), (21), (22), (25), (27) provide us with

$$X_{VIII} \equiv H_{III} = \left(B_{III}\,\frac{R}{\dot{R}},\, 0,\, 0,\, 1\right) \qquad \text{condition } \ddot{R} = 0 \tag{35}$$

For $H_{IV} = \left(H^t,\, 0,\, H^\theta,\, H^\phi\right)$, (18) and (22) fix H^t. As for the angular components, there exist two linearly independent solutions, namely, $H^\theta(\phi) = \sin\phi$ and $H^\theta(\phi) = \cos\phi$. We, therefore, obtain

$$X_{IX} \equiv H_{IV} = \left(B_{IV}\,\frac{R}{\dot{R}},\, 0,\, \sin\phi,\, \cos\phi\cot\theta\right) \qquad \text{condition } \ddot{R} = 0 \tag{36}$$

$$X_X \equiv H_V = \left(B_V \frac{R}{\dot{R}},\, 0,\, \cos\phi,\, -\sin\phi\cot\theta\right) \qquad \text{condition } \ddot{R} = 0 \qquad (37)$$

where B_I, B_{II}, B_{III}, B_{IV}, B_V are constants.

The vector H_I is always spacelike, H_{II} is always timelike and H_{III}, H_{IV} and H_V can be spacelike, null or timelike depending on the values of corresponding constants. The conformal factor in each of the above five cases is equal to the corresponding constant B_J, $(J = I, II, III, IV, V)$.

3.3 Special CKVs

There are *no* Special Conformal Killing vectors for the metric (1). The conditions $\Phi_{,ij} = 0$, together with the equations (18) - (27) force $\Phi =$ constant.

3.4 Non-special CKVs

For Non-Special Conformal Killing vectors $\Phi_{,ij} \neq 0$.

In this case we find two vectors given by

$$X_{XI} = N_I = R\frac{\partial}{\partial t} \qquad (38)$$

$$X_{XII} = N_{II} = R\log y\frac{\partial}{\partial t} + \frac{y'}{y}\int\frac{dt}{2R}\frac{\partial}{\partial r} \qquad (39)$$

with conformal factors given by

$$\phi = \dot{R} \qquad (40)$$

$$\phi = \dot{R}\log y + \int\frac{dt}{2R} \qquad (41)$$

respectively.

4 Summary

It should be noted that out of the twelve CKVs given above only the X_I, X_{II}, X_{III}, X_{VI}, X_{XI}, X_{XII} are linearly independent. Other conformal Killing vectors reported earlier are obtainable from these six vectors under the stated conditions.

In what follows, we summarize the conformal factors for these six vectors. We also provide in this section the Lie Brackets for these six vectors.

The conformal factors for these CKVs are as follows:

CKV	$\rightarrow$	X_I/K_I	X_{II}/K_{II}	X_{III}/K_{III}
Φ	$\rightarrow$	0	0	0
CKV	$\rightarrow$	X_{VI}/H_I	X_{XI}/N_I	X_{XII}/N_{II}
Φ	$\rightarrow$	B_I	$\dot{R}$	$\dot{R}\log y + \int \frac{dt}{2R}$

The above CKVs[2] form the basis of the Lie Algebra of CKVs for the metric (1). The Lie Algebra of CKVs is therefore six dimensional, that is, G_6. The Lie brackets and corresponding structure constants are easily obtained as follows:

$$\begin{aligned}
[K_I, K_{II}] &= K_{III} \\
[K_{II}, K_{III}] &= K_I \\
[K_{III}, K_I] &= K_{II} \\
[H_I, N_{II}] &= N_I \\
[N_I, N_{II}] &= \frac{1}{2} H_I
\end{aligned}$$

We note that the three Killing vectors - K_I, K_{II}, K_{III} - form a sub-algebra corresponding to the rotational symmetry of the space-time. On the other hand, the remaining three Conformal Killing Vectors - H_I, N_I, N_{II} - form another sub-algebra isomorphic to the Lie Algebra corresponding to the group $E^+(2)$ [8]. This is realized by setting $U_1 = H_I, U_2 = -\sqrt{2}\, i\, N_I, U_3 = \sqrt{2}\, i\, N_{II}$. This is an interesting aspect of the space-time of the metric (1).

[2]Note: The above results have been confirmed using the differential equation solver within `PROGRAM LIE` [7].

5 Geodesic equations of motion

The Lagrangian for the metric (1) is

$$2\,\mathcal{L} \;=\; -\,y^2\,\tilde{t}^2 \;+\; R^2\,\Big[\,2\,(y')^2\,\tilde{r}^2 \;+\; y^2\,\tilde{\theta}^2 \;+\; y^2\,\sin^2\theta\,\tilde{\phi}^2\Big] \tag{42}$$

where an overhead $\tilde{}$ denotes a derivative with respect to the affine parameter s along the geodesic. Further, $\mathcal{L}$ takes values 0 for null, $+1$ for spacelike and -1 for timelike geodesics.

The geodesic equations of motion are obtained from the Euler-Lagrange equations:

$$\frac{d}{ds}\left(\frac{\partial(2\mathcal{L})}{\partial\dot{x}^a}\right) \;=\; \frac{\partial(2\mathcal{L})}{\partial x^a} \tag{43}$$

In what follows, we will use (42) to reduce the terms after rearrangement.

The r-equation is obtained as:

$$\frac{d}{ds}\left(R^2\,y\,y'\,\tilde{r}\right) \;=\; \mathcal{L} \tag{44}$$

and can be easily integrated to obtain

$$\tilde{r} \;=\; \frac{\mathcal{L}\,s \,+\, k_1}{R^2\,y\,y'} \tag{45}$$

The ϕ-equation is

$$\frac{d}{ds}\left(R^2\,y^2\,\sin^2\theta\,\tilde{\phi}\right) \;=\; 0 \tag{46}$$

and it implies

$$\tilde{\phi} \;=\; \frac{k_2}{R^2\,y^2\,\sin^2\theta} \tag{47}$$

where k is a constant of integration.

The θ-equation is obtained as

$$\frac{d}{ds}\left(R^2\,y^2\,\tilde{\theta}\right) \;=\; R^2\,y^2\,\sin\theta\,\cos\theta\,\tilde{\phi}^2 \tag{48}$$

On using (47) and rearranging terms, we obtain

$$\frac{d}{ds}\left[\left(R^2\,y^2\,\tilde{\theta}\right)^2 \;+\; k_2^2\cot^2\theta\right] \;=\; 0 \tag{49}$$

which implies

$$\left(R^2\,y^2\,\tilde{\theta}\right)^2 \;+\; k_2^2\cot^2\theta \;=\; k_3$$

where k_3 is an integration constant. Hence,

$$\tilde{\theta} \;=\; \frac{\pm 1}{R^2\,y^2}\sqrt{k_3 \;-\; k_2^2\,\cot^2\theta} \tag{50}$$

The t-equation is obtainable as

$$\frac{d}{ds}\left(R\,y^2\,\tilde{t}\right) \;=\; -\,2\,\mathcal{L}\,\dot{R} \tag{51}$$

where we have used (42) to reduce terms.

It is difficult to integrate this equation directly. However, the solution is obtainable from (42) using (45), (47), (48) as

$$\tilde{t} \;=\; \sqrt{-\frac{2\mathcal{L}}{y^2} \;+\; \frac{1}{y^4R^2}\left[2\,(\mathcal{L}s \;+\; s)^2 \;+\; k_3 \;+\; k_2^2\right]} \tag{52}$$

Therefore, the geodesic of the metric (1) has the tangent vector

$$T^a \;=\; \left(\tilde{t},\,\tilde{r},\,\tilde{\theta},\,\tilde{\phi}\right) \tag{53}$$

with $\tilde{t}$, $\tilde{r}$, $\tilde{\theta}$, $\tilde{\phi}$ as obtained above.

Of course, for null geodesics, we obtain by direct integration of (51):

$$Ry^2\tilde{t} \;=\; \delta \tag{54}$$

where δ is a constant. Therefore, in general, the null geodesic of the metric (1) has the tangent vector

$$n^a \;=\; \frac{1}{y^2\,R^2}\left(\delta\,R,\;\frac{k_1\,y}{y'},\;\pm\sqrt{k_3 \;-\; k_2^2\cot^2\theta},\;\frac{k_2}{\sin^2\theta}\right) \tag{55}$$

with the different constants satisfying a relation obtainable from $n^a\, n_a \,=\, 0$. In particular, for the *radial null geodesic* we have

$$\delta \;=\; \pm\sqrt{2}\, k_1 \tag{56}$$

Along a null geodesic $x^a(s)$ with tangent vector $n^a \,=\, dx^a/ds$, the following holds:

$$n^a{}_{;\,b}\, n^b \;=\; 0 \;=\; n^a\, n_a \tag{57}$$

Any conformal Killing vector, $\mathbf{X}$, generates along a null geodesic a constant of the motion:

$$\frac{d}{ds}\,(\,\mathbf{X} \bullet \mathbf{n}\,) \;=\; X_{a;\,b}\, n^a\, n^b \;=\; 2\,\Phi\, g_{ab}\, n^a\, n^b \;=\; 0 \tag{58}$$

Each geodesic is specified by six parameters: four to determine a point on it and two to determine the direction of its tangent. Thus, the general solution of the null geodesic equation for the metric (1) depends on six functionally independent constants of null geodesic motion. These constants of motion are obtainable from the six conformal Killing vectors and the above tangent to the null geodesic trivially.

6 Discussion

Most of the known solutions of the Einstein field equations admit some nontrivial isometry group. Further, there are many solutions with conformal symmetry, but in most of these the symmetry is actually homothetic. Much physical insight on astrophysical and cosmological questions has been obtained from the study of such solutions. Of course, with the assumption of the spherical symmetry, there is the rotational isometry group associated with the spacetime. The question then is of obtaining further symmetries of the spherically symmetric spacetime. In this respect, our spacetime here is interesting since it has only one homothetic symmetry but two conformal symmetries. A homothetic Killing vector leads to self-similarity while scaling all the distances by the same constant factor. The conformal symmetries provide the constants of geodesic motion along null trajectories of massless particles.

In this paper, we obtained all the conformal Killing vectors for the metric (1) and their Lie Algebra. We showed that, for the space-time of the metric (1), there exists a sub-algebra isomorphic to the Lie Algebra of $E^{+}(2)$. This Lie Algebra has an associated Local

Lie Transformation Group $E^+(2)$ that is a group of distance preserving transformations in the plane R_2.

We also obtained explicitly the tangent vector to the geodesics of the metric (1). In particular, for the trajectories of massless particles or the null geodesics, the constants of motion can be obtained from the CKVs presented. The results obtained here are of importance for the study of astrophysical problems involving stellar models or for the study of inhomogeneous cosmological models based on the metric (1).

Acknowledgements

KSG thanks the University of Natal and the National Research Foundation for ongoing support. He also thanks CIRI for their kind hospitality during the course of this work.

References

[1] Wagh, S. M., Govender, M., Govinder, K. S., Maharaj, S. D., Muktibodh, P. S. and Moodley, M. (2001) *Class. Quantum Grav.* **18**, 2147 - 2162.

[2] Wagh, S. M., Saraykar, R. V., Muktibodh, P. S. and Govinder, K. S. (2001) *Gen. Rel. Grav.* submitted.

[3] Wagh, S. M. (2001) to be submitted.

[4] Robertson, H. P. and Noonan, T. W. (1969) *Relativity and Cosmology* (W. B. Saunders Co., London).

Choquet-Bruhat, C., Dewitte-Morette, C. and Dillard-Bleick, M. (1977) *Analysis, Manifolds and Physics* (North-Holland, Amsterdam).

Kramer, D., Stephani, H., MacCallum, M. A. H. and Herlt, E. (1980) *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge).

[5] **Note:** In an unpublished doctoral thesis entitled *Conformal Motions in General Relativity* (University of Natal, Durban, South Africa, 1997) Dr. S. Moopanar has worked out the most general form of the solution of the conformal Killing equations for spherically symmetric spacetimes. There are eleven integrability conditions to be satisfied by the general solution. Our results for the metric (1) can be obtained by solving the integrability conditions listed by Dr. Moopanar.

[6] Maartens, R. and Maharaj, S. D. (1986) *Class. Quantum Grav.*, **3**, 1005-1011.

[7] Head, A. K. (1993) *Comp. Phys. Comm.*, **77**, 241-248.

[8] Miller, W. (1972) *Symmetry Groups and Their Applications* (Academic Press, New York) p. 190.